

# Physical Cosmology Exercises

Phil Bull, June 8, 2026

## Purpose of exercises

These exercises were set on a weekly basis. Each week has a set of practise questions, with an ‘assessed question’ at the end that consolidates the material in the previous questions (and which was used as a basis for weekly summative assessment). There is also a ‘tutorial question’ for each week, which was attempted in a 1-hour in-person exercise class.

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## Exam question pro-tips

- Show your working;
- Make it clear what your final answer is (e.g. underline it);
- Write neatly and legibly;
- Diagrams/sketches should be neat and properly labelled, but don’t need to be super precise (no need for a ruler);
- Don’t forget units;
- Use maths symbols (e.g. equals sign, integrals) properly;
- Cross-out wrong answers simply but clearly;
- If a question asks you to explain or discuss something, write a few concise sentences (not just a handful of words, and not a long essay!);
- **Read the instructions fully and carefully before you start writing!**

## Section 1: Expanding universe

### 1. Maths practice: Unit conversions

- What is the frequency, in GHz, of electromagnetic radiation with a wavelength of 21cm?
- A typical cluster of galaxies has a mass of around  $10^{14} M_{\odot}$ . What is this in kg?
- What is 6 arcsec. in degrees? How many radians are in a degree?
- Consider a neutrino of energy 0.06 eV in its rest frame. What is its mass in kg?

### 2. History of the Universe

- Give a one-sentence description of what happened during each of the following epochs:  
(i) Inflation; (ii) Structure formation; (iii) Cosmic Dawn.
- Put the following events in chronological order (from earliest to most recent):  
*Extinction of the dinosaurs — Universe became transparent to light — Stoke City won the League Cup — First neutral atoms formed — Inflation — Formation of first stars and galaxies — First nuclei formed — Large-scale structure formed.*

### 3. Spectroscopy

An emission line from a galaxy is emitted with a rest-frame wavelength of 656.282 nm, and is detected on Earth at a wavelength of 748.818 nm.

- What is the redshift of the galaxy?
- Interpret this redshift as a Doppler shift. Use this to calculate the recession velocity of the galaxy,  $v_{\text{rec}}$
- A second galaxy of the same type is found at a redshift  $z = 1.41$ . What wavelength will the emission line be observed at for this galaxy?
- Calculate the recession velocity for this galaxy. Comment on the physical interpretation of this velocity.

### 4. Redshift

- By considering how the wavelength of a light ray changes after space has expanded by some factor, show that the scale factor is related to the cosmological redshift by  $a \propto (1+z)^{-1}$ .
- What redshift corresponds to when the Universe was (i) half its current size; (ii) a tenth its current size?
- The Lyman- $\alpha$  ( $n = 2 \rightarrow 1$ ) emission line of hydrogen is emitted at a wavelength of 121.6 nm in the rest frame. Galaxies A and B are observed with Lyman- $\alpha$  emission lines at frequencies of  $0.99 \times 10^{15}$  Hz and  $1.90 \times 10^{15}$  Hz respectively. (i) What is the redshift of each galaxy? (ii) Which galaxy is further from us?

### 5. Olbers' paradox

By drawing a diagram, explain Olbers' Paradox. How does the observation that the Universe is expanding solve the paradox? Name one other possible solution and explain why it does not work.

#### Assessed question: Galaxy redshifts

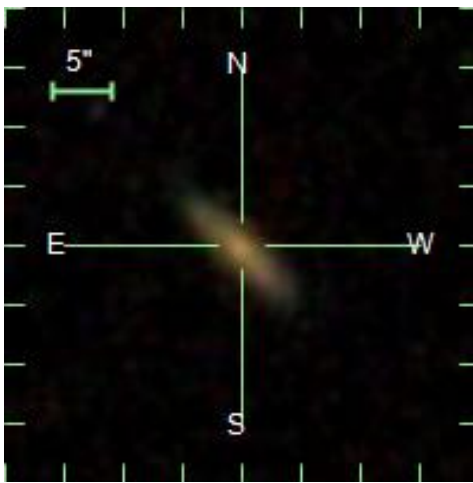
Three galaxies are observed in an emission line with rest-frame wavelength  $\lambda_{\text{em}}$ . Galaxy A is observed at a wavelength  $\lambda_A = 396.9$  nm, galaxy B at  $\lambda_B = 392.4$  nm, and galaxy C at  $\lambda_C = 798.3$  nm.

- Galaxy A is found to have a peculiar velocity of +304 km/s in the direction away from the observer, and an observed redshift of  $z_{\text{obs}} = 0.009$ . Use this information to calculate  $\lambda_{\text{em}}$  in nm.
- Calculate the observed redshift of galaxy B. What can you infer about the peculiar velocity of this galaxy?
- Calculate the observed recession velocity of galaxy C as a fraction of the speed of light. If this galaxy has a peculiar velocity of  $-230$  km/s, what is its cosmological redshift?
- Which of the galaxies was the first to emit the light that we now observe?

## Tutorial question 1: Spectrometry of a spiral galaxy

The Sloan Digital Sky Survey took images and frequency spectra of several million astronomical objects during its operations from 2000-2020. We will analyse the data for a single object – a spiral galaxy.

- Use the information available to identify emission and absorption lines in the observed spectrum and measure their observed wavelengths,  $\lambda_{\text{obs}}$ . N.B. Some lines will be too faint to see in amongst the noise. There are also some spurious spikes in the data that do not correspond to line emission.
- Use your measurements to calculate the observed redshift,  $z$ , of the galaxy. By reinterpreting this as a Doppler shift, also calculate the recession velocity,  $v_{\text{rec}}$ , in km/s.
- Using the Hubble law with a proportionality constant  $H_0 \approx 70$  km/s/Mpc, estimate the distance to this galaxy in Mpc. Approximately how long ago was the light we observe from this galaxy emitted?
- Calculate the approximate diameter of the spiral galaxy, in kpc.



Line	$\lambda_{\text{em}}$ (Angstrom)	Type
[N II]	5755	Emission
He I	5876	Emission
Na I D1	5889	Absorption (doublet)
Na I D2	5895	Absorption (doublet)
[O I]	6300	Emission
[S III]	6312	Emission
[O I]	6363	Emission
[N II]	6548	Emission
H $\alpha$	6563	Emission / absorption
[N II]	6583	Emission
[S III]	6716	Emission
[S III]	6730	Emission

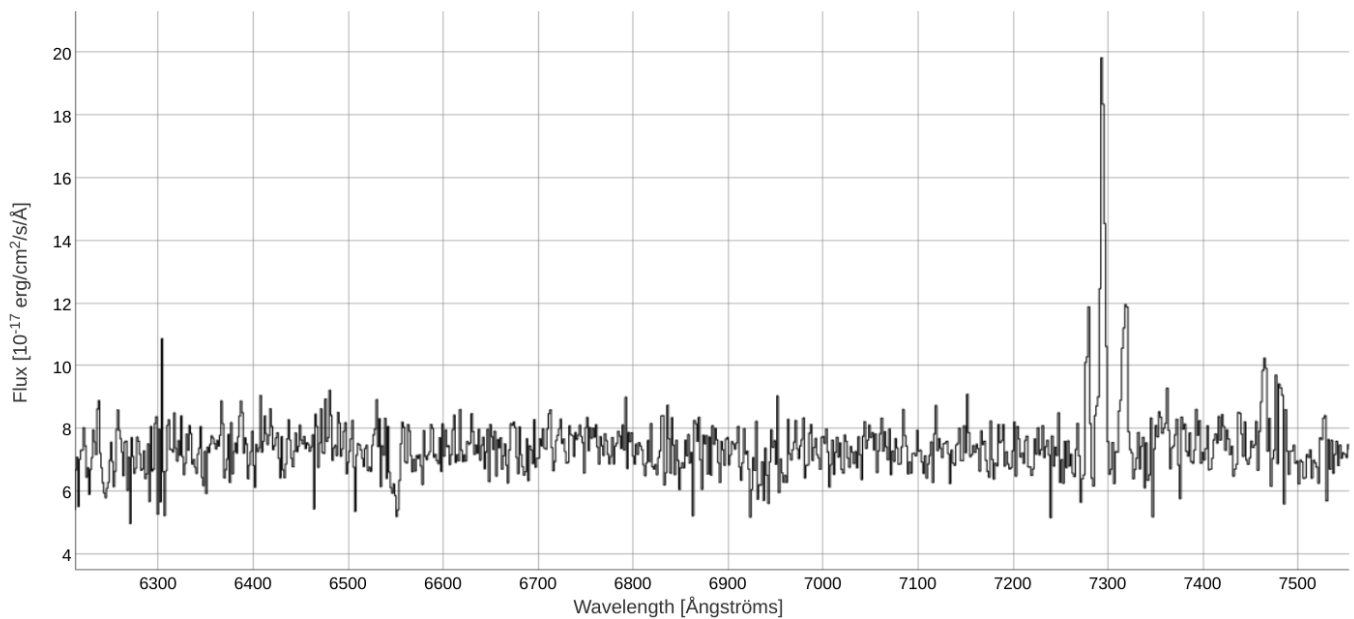


Figure 1: *Upper left:* Multi-band optical image of a galaxy observed by the Sloan Digital Sky Survey. *Upper right:* Table of common emission and absorption lines and their rest-frame wavelengths in the relevant part of the electromagnetic spectrum. *Bottom:* Section of the frequency spectrum of this galaxy measured by SDSS. (Credit: SDSS)

## Section 2: Geometry and distance

### 1. Maths practice: Taylor expansions and trigonometry

- Consider a right-angled triangle with hypotenuse of length 20, and another side of length 4 opposite an angle  $\alpha$ . (i) Find the length of the other side, adjacent to  $\alpha$ . (ii) Find the remaining angle,  $\beta$ , in degrees.
- Expand  $y = (1 + 2x)^{-2}$  to linear order about  $x = 0$ .
- Expand  $y = \ln(1 + x)$  to linear order about  $x = 1$ .
- Expand  $y = \sin(3x)$  to quadratic order about  $x = 0$ .

### 2. Small-angle approximation

In cosmology, we are often dealing with small angles,  $\theta \simeq 0$ . This allows us to use the *small-angle approximation* to simplify some expressions.

- In the small angle approximation, show that  $\tan \theta \simeq \theta$ .
- By evaluating the next-to-leading-order term in the Taylor expansion, find the angle at which the small angle approximation to  $\tan \theta$  differs from the exact solution by (approximately) more than 10%.

### 3. Angles, distances, and velocities

- Explain the difference between parallax angle and the angular size of an object.
- Explain the difference between proper and comoving coordinates.
- If an object stays at an exactly constant comoving distance from us as the Universe expands, what does this say about its peculiar velocity?

### 4. Curved universe

Consider an ant walking around on the surface of a beach ball.

- Using this as an analogy, explain what is meant by a spatially-closed universe.
- What would be equivalent analogies for spatially flat and spatially open universes?

### 5. Metric tensor

Consider a space-time with the following metric tensor, for coordinates  $(t, x, y, z)$ :

$$g_{ab} = \begin{pmatrix} -c^2 & & & \\ & a_{\parallel}^2(t) & & \\ & & a_{\perp}^2(t) & \\ & & & a_{\perp}^2(t) \end{pmatrix}.$$

- Write down the line element,  $ds^2$ , for this space-time.
- What conditions must be satisfied for this to be an FLRW metric? (Hint: Consider the normalisation and rate of change of the two different scale factors,  $a_{\parallel}$  and  $a_{\perp}$ .)

### 6. Conformal factor

Two metrics,  $g$  and  $\tilde{g}$ , are said to be conformally equivalent if they satisfy a relation of the form  $\tilde{g}_{ab} = f(\vec{x}, t)g_{ab}$ , where  $f(\vec{x}, t)$  is the *conformal factor*. Note how  $f$  is a scalar quantity, i.e. it is the same for all indices  $a$  and  $b$ .

- The Minkowski metric of Special Relativity is given by  $\eta_{ab} = \text{diag}(-c^2, 1, 1, 1)$ . By first performing an appropriate coordinate transformation, show that any flat FLRW metric is conformal to Minkowski.
- Conformal transformations preserve angles (so angles in a spacetime described by metric  $g$  remain the same after transforming to metric  $\tilde{g}$ ). Use this fact to explain why a *closed* FLRW spacetime can't be conformally equivalent to Minkowski.

**Assessed question: Coordinates in an expanding universe**

Consider the following *comoving* positions: Galaxy A is at  $\vec{x}_A = (100, 0, 0)$  Mpc, Galaxy B is at  $\vec{x}_B = (0, 40, 0)$  Mpc, and the observer is at the origin.

- (a) Calculate the proper distance between the observer and Galaxy A when the scale factor  $a = 0.6$ . Give your answer in Mpc.
- (b) The present expansion rate in this universe is  $H_0 = 74$  km/s/Mpc. What are the cosmological recession velocities of Galaxies A and B as seen by the observer today (when  $a = 1$ )? Give your answer in km/s.
- (c) The Hubble Law can be used to show that the relation between distance and redshift is approximately  $d \approx cz/H_0$ . This relation only holds at low redshifts, when  $z \approx 0$ .  
Use this relation to calculate the cosmological redshifts,  $z$ , of Galaxies A and B as seen by the observer.
- (d) Now, calculate the cosmological redshifts of Galaxy B and the observer as seen from Galaxy A.

## Tutorial question 2: Parallax distances and the Hubble Law

Doubly-ionised Oxygen (O[III]) has an emission line with a rest-frame wavelength of 166.59 nm. The table below shows the measured wavelengths of the O[III] line from several galaxies, along with their parallax in micro-arcseconds. (We will assume that measurement errors are negligible.)

<b>Galaxy</b>	$\lambda_{\text{obs}}^{\text{O[III]}}$ (nm)	<b>Parallax</b> ( $\mu\text{as}$ )
A	167.090	0.0634
B	167.562	0.0462
C	167.699	0.0369
D	168.472	0.0221

- Calculate the observed redshift and recession velocity of each galaxy.
- Calculate the parallax distance to each galaxy (in pc).
- Plot these quantities on a graph (draw it by hand). Add a straight line that fits the data points as well as possible and measure its gradient.
- Using this plot, infer the value of the Hubble parameter,  $H_0$ , in units of km/s/Mpc.
- Do the points on the graph follow a perfectly straight line? If not, explain why.
- ESA's Gaia space telescope has measured the parallaxes of 1.3 billion objects with a precision of up to 0.04 mas (milli-arcsec) for the brightest objects.

Using your measurement of  $H_0$  from part (d), calculate the maximum redshift of objects that Gaia can reliably measure the parallax distances of.

## Section 3: Friedmann equation

### 1. Maths practice: ordinary differential equations

- Solve  $dx/dt = \pm x^{-\frac{3}{2}}$  when  $x(t_0) = 1$ .
- Solve  $d^2y/dt^2 = \sqrt{\lambda}y$  to find  $y(t)$ .
- Solve  $p dp = \sqrt{p^2 + \kappa p^3} du$  to find  $u(p)$ .

### 2. Unit conversions and density parameters

Take a universe with  $H_0 = 70$  km/s/Mpc.

- What is the critical density today in units of (i)  $\text{kg m}^{-3}$  and (ii)  $M_\odot \text{Mpc}^{-3}$ ?
- Earth is approximately 8 kpc from the centre of the Milky Way. How many lightyears is that?
- Imagine an object of length 3m is expanding with the Hubble flow. How much length will it gain after a year? What is the length gain as a fraction of its original length?
- Repeat exercise (c) for an object of length 10 Mpc.

### 3. Spatial curvature

Consider a universe with  $H_0 = 67$  km/s/Mpc,  $\Omega_m = 0.9$ , and  $\Omega_k = 0.1$ .

- Is this universe flat, open, or closed?
- Sketch roughly how a triangle would appear in this space. Do the angles of the triangle add up to  $180^\circ$ ?
- Describe the ultimate fate of this universe, far into its future.
- How many protons would you need to add/remove per  $m^3$  of space at  $t = t_0$  to make this universe flat?

### 4. Matter-dominated universe

Consider a flat, expanding universe filled only with pressureless matter, and a Hubble expansion rate today of  $H_0 = 75$  km/s/Mpc.

- Write down the Friedmann equation for this universe.
- What is the critical density at the present day in this universe, in units of  $M_\odot \text{Mpc}^{-3}$ ?
- Solve for the scale factor,  $a(t)$ . What is the present age of this universe?

### 5. Expansion rate $\blacktriangle$

Consider a universe containing only matter and a cosmological constant. The expansion rate at three redshifts is given by:  $H(z = 0.3) = 73.62$  km/s/Mpc;  $H(z = 0.8) = 97.16$  km/s/Mpc;  $H(z = 1.3) = 128.24$  km/s/Mpc.

Use these data to calculate the values of  $\Omega_m$ ,  $\Omega_\Lambda$ , and  $H_0$ . Is this universe open, closed, or flat?

### 6. Time derivative of the Friedmann equation $\blacktriangle$

- Show that  $d(H^2)/dt = 2H\dot{H}$ , where the dot denotes a time derivative.
- Derive an expression for the deceleration parameter,  $q(a) = -(1 + \dot{H}/H^2)$ , in a universe with matter and radiation fractions today of  $\Omega_m$  and  $\Omega_r$  respectively.
- Use your expression to calculate  $q(t_0)$  in a matter-only and a radiation-only universe respectively.

### Assessed question: Radiation-dominated universe

Consider a universe containing only radiation, with unknown curvature.

- The energy density of radiation at scale factor  $a = 0.04$  is  $\rho_r(a = 0.04) = 1.7 \times 10^{-19} \text{ kg m}^{-3}$ . Calculate the energy density of radiation at  $a = 1$ , in the same units.
- The expansion rate today is  $H_0 = 440$  km/s/Mpc. Use this to calculate the curvature parameter,  $\Omega_k$ .
- Calculate the value of the scale factor  $a$  when the expansion rate is  $H(a) = 267$  km/s/Mpc.

### Tutorial question 3: Applications of the Friedmann equation

The Friedmann equation is given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3},$$

where dots denote a time derivative.

- (a) Write down an expression for the critical density today,  $\rho_{\text{cr},0}$ .
- (b) Divide the Friedmann equation by  $H_0^2$ . By including appropriate scale factor-dependent terms for matter, radiation, curvature, and a cosmological constant, use your result to prove that  $\Omega_m + \Omega_r + \Omega_k + \Omega_\Lambda = 1$ .
- (c) Explain how the result above allows us interpret the  $\Omega$  parameters as fractions (fractions of what?).
- (d) For a fixed value of  $H_0$ , put the following (spatially-flat) universes in *increasing* order of age: matter-only; radiation-only; Cosmological Constant-only; 95% matter and 5% radiation at  $t_0 = \text{today}$ .
- (e) What is the redshift of matter-radiation equality (i.e. the redshift at which  $\rho_m = \rho_r$ ) in the 95% matter + 5% radiation universe from part (c)?

## Section 4: Distances and horizons

### 1. Maths practice: Maxima and minima of functions

- Find the extremum (max. or min. value) of the function  $y = x^2$ . Is this a maximum or a minimum?
- Find the extrema of the function  $y = 4x^2 + 2x - 3$ .
- Find the extrema of the function  $y = x^2 + 3e^{-x^2}$ .

### 2. Etherington distance-duality relation

The Etherington distance-duality relation is the statement that  $d_L(z) = (1+z)^2 d_A(z)$ .

- Calculate the Taylor expansion of the luminosity distance,  $d_L(z)$ , up to second order about  $z = 0$ . Assume an arbitrary expansion rate  $H(z)$  (i.e. don't substitute in the Friedmann equation for  $H(z)$ ).
- Derive a similar Taylor expansion for the angular diameter distance,  $d_A(z)$ .
- Compare the first-order parts of the two Taylor expansions. What do you notice?

### 3. Angular diameter distance

- If a galaxy at redshift  $z = 3.83$  lies at an angular diameter distance of  $d_A = 1460$  Mpc, what is its luminosity distance (in Mpc)? What is the comoving distance travelled by light from this galaxy?
- A galaxy of proper (physical) diameter 41 kpc is observed to subtend an angle of 1.8 arcsec. What is the angular diameter distance to the galaxy?

### 4. Horizons in a matter-only universe

Consider a flat, matter-only universe with an expansion rate today  $H_0$ .

- Derive an expression for the Hubble horizon,  $r_{\text{HR}}(a)$ , in this universe.
- Derive an expression for the particle horizon,  $r_{\text{H}}(a)$ , in this universe.
- Sketch a graph showing how  $r_{\text{HR}}$  and  $r_{\text{H}}$  depend on scale factor.

### 5. Scale factor in a closed universe $\blacktriangle$

The scale factor in a closed universe, containing only matter and positive curvature, is given by the following parametric solution:

$$a(\tau) \propto (1 - \cos \tau); \quad t(\tau) \propto (\tau - \sin \tau),$$

where  $\tau$  is the conformal time.

- Show that this is indeed a solution to the Friedmann equation in a closed universe with matter.
- Derive the constants of proportionality for  $a(\tau)$  and  $t(\tau)$  in terms of  $H_0$ ,  $\Omega_m$ , and  $\Omega_k$ .
- Plot the scale factor,  $a(t)$ , and expansion rate,  $H(t)$ , for this universe, as a function of time,  $t$ .
- At what time  $t$  does the scale factor reach its maximum?

#### Assessed question: Distance in a matter-dominated universe

Consider a flat, expanding universe containing only matter, with an expansion rate today of  $H_0 = 60$  km/s/Mpc.

- Calculate the cosmological redshift,  $z$ , of a galaxy that lies a comoving distance of 480 Mpc away from the observer.
- The angular diameter of the galaxy is 24.7 arcsec. What is its physical diameter, in kpc?
- A supernova of peak luminosity  $L = 7 \times 10^9 L_\odot$  explodes in the galaxy.  
A telescope with a flux sensitivity (minimum observable flux) of  $f > 20 L_\odot \text{Mpc}^{-2}$  is pointed at the galaxy. Calculate whether the telescope will be able to see the supernova explosion.

## Tutorial question 4: Maximum of the angular diameter distance

Consider a flat, matter-only universe with Hubble parameter  $H_0$ .

- (a) Write down the definition of the angular diameter distance,  $d_A(z)$ , in terms of the angular size  $\Delta\theta$  and physical size  $d$  of a distant object observed at redshift  $z$ . How is the angular diameter distance related to the comoving distance to the object?
- (b) Derive a mathematical expression for  $d_A(z)$ , the angular diameter distance as a function of redshift, in this universe.
- (c) Calculate the redshift at which the angular diameter distance reaches its maximum value.
- (d) Consider an object of fixed physical size  $d$ . Explain how its angular size,  $\Delta\theta$ , would change depending on what redshift it was observed at in this universe (hint: sketch a graph).
- (e) What are the implications of part (d) for the angular sizes of objects that we see at great distances?

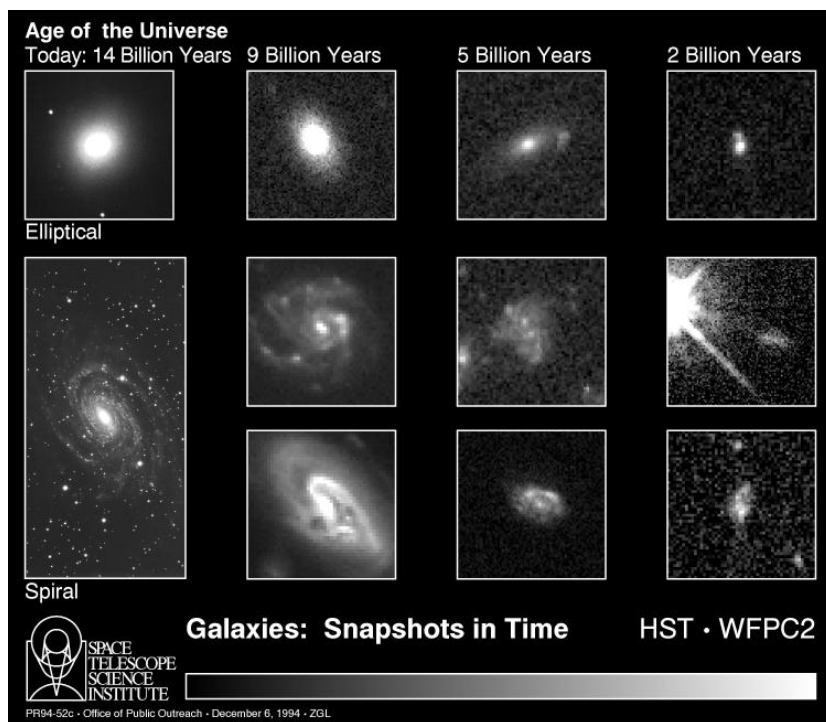


Figure 1: Images of galaxies at increasing distances (i.e. looking progressively further back in time), taken by the Hubble Space Telescope (Credit: HST/STScI).

## Section 5: Cosmic acceleration

### 1. Equation of state of radiation

Using only the conservation equation and the fact that  $\rho_r \propto a^{-4}$ , calculate the equation of state  $w$  for (non-interacting) radiation in a universe that contains matter, radiation, and a cosmological constant.

### 2. Deceleration parameter

The scale factor can be written as a Taylor expansion about  $t = t_0$ :

$$a(t) \approx 1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2 (t - t_0)^2 + \dots$$

(a) Using the definition of the deceleration parameter from the notes,

$$q(a) = - \left( \frac{a}{\dot{a}} \right)^2 \frac{\ddot{a}}{\dot{a}},$$

and the Taylor expansion above, show that  $q(a) = q_0$  at  $t = t_0$ .

(b) Use the same definition of  $q(a)$  from part (a) above to show that

$$q(a) = - \left( 1 + \frac{\dot{H}}{H^2} \right).$$

### 3. Onset of acceleration

Consider a flat universe with  $\Omega_m = 0.3$ ,  $\Omega_\Lambda = 0.7$ , and  $H_0 = 60$  km/s/Mpc. Use the Raychaudhuri equation to find the scale factor at which the expansion of this universe first starts to accelerate ( $\ddot{a} > 0$ ).

### 4. de Sitter universe

In the far future, our Universe is expected to asymptotically approach a de Sitter spacetime, which is spatially flat and has  $\Omega_\Lambda = 1$ .

(a) Write down the Friedmann and Raychaudhuri equations for a flat,  $\Lambda$ -only universe.

(b) Find a solution for  $a(t)$  in this universe.

(c) Derive an expression for the expansion rate,  $H(a)$ , in this universe.

(d) By using the conservation equation, show that  $w = -1$  in this universe.

(e) Find an expression for the Hubble radius,  $r_{\text{HR}}(a)$ , in this universe as a function of scale factor. Sketch a graph to show how it depends on  $a$ .

### 5. Arbitrary equation of state $\blacktriangle$

Consider a type of matter/energy that has an arbitrary equation of state that depends on scale factor,  $w(a)$ .

(a) We will approximate  $w(a)$  using a Taylor expansion about  $a = 1$ . Find an appropriate Taylor expansion to linear order in the scale factor. Your expansion should reduce to the following:

$$\begin{aligned} \lim_{a \rightarrow 1} w(a) &= w_0 \\ \lim_{a \rightarrow 0} w(a) &= w_\infty \equiv w_0 + w_a, \end{aligned}$$

where  $w_0$  and  $w_a$  are coefficients to be specified.

(b) Using only the conservation equation and your approximate expression for  $w(a)$ , find an expression for the energy density,  $\rho(a)$ , of this type of matter/energy as a function of scale factor,  $a$ .

(c) Draw a rough sketch of the energy density as a function of scale factor,  $\rho(a)$ , for the following values of  $(w_0, w_a)$ :  
 $(-1, 0)$ ,  $(-1, 0.1)$ ,  $(-1, -0.1)$ .

(d) So-called 'phantom' dark energy has an equation of state  $w < -1$  at all times. Using your result from part (c), explain how phantom dark energy would affect the expansion rate far into the future.

(e) What would happen if  $w = +1$ ?

## Tutorial question 5: Marginally-accelerating universe

Consider a flat, expanding universe that is neither accelerating nor decelerating, i.e.  $\ddot{a} = 0$ , and that is filled with only a single kind of matter/energy with a constant equation of state parameter  $w$ .

- (a) Find the value of  $w$  that is needed to produce this result.
- (b) Derive an expression for the energy density of this kind of matter/energy as a function of scale factor,  $\rho(a)$ .
- (c) Sketch the energy density as a function of scale factor.
- (d) ESA's Euclid space telescope is expected to launch in 2022, and will be able to measure the equation of state of dark energy with a precision of  $\sigma_w = 0.017$ , where  $\sigma_w$  is the anticipated standard deviation of the measurement. Will Euclid be able to tell the difference between marginal acceleration vs acceleration caused by a non-zero Cosmological Constant?

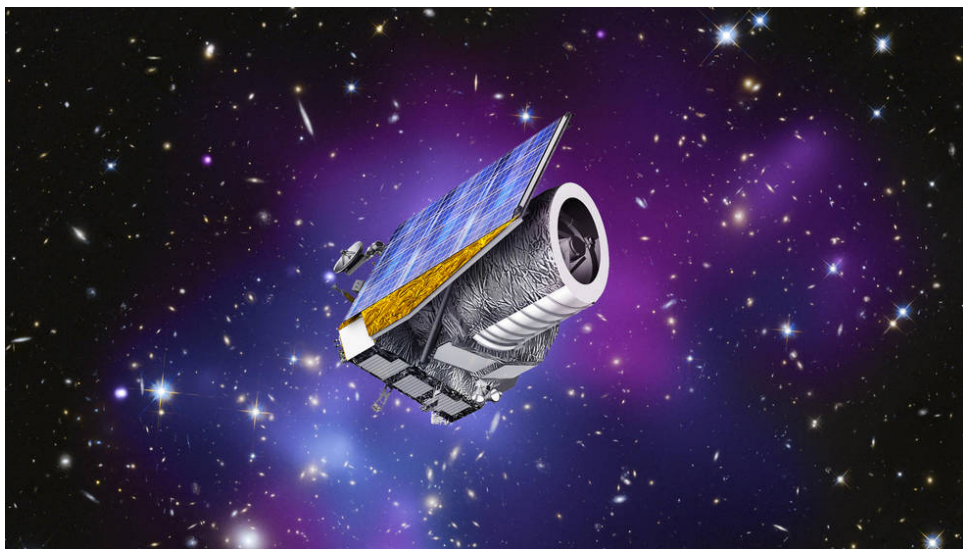


Figure 1: An artist's impression of ESA's Euclid spacecraft (Credit: ESA/C. Carreau).

## Sections 6 & 7: Big Bang Nucleosynthesis and CMB radiation

### 1. Maths practice: Change of variables

- (a) By writing  $x = \nu/\nu_0$ , convert the integral  $I = \int \nu^\beta d\nu$  into a dimensionless integral times a dimensionful prefactor. What are the dimensions of the prefactor?
- (b) The first few Legendre polynomials can be written as

$$\mathcal{P}_0(\theta) = 1; \quad \mathcal{P}_1(\theta) = \cos \theta; \quad \mathcal{P}_2(\theta) = \frac{1}{2} (3 \cos^2 \theta - 1).$$

By performing the substitution  $\mu = \cos \theta$  and then integrating, show that these polynomials satisfy an *orthogonality relation*,

$$\int_0^\pi \mathcal{P}_n(\theta) \mathcal{P}_m(\theta) \sin \theta d\theta = 0 \quad \text{when } n \neq m.$$

- (c) By using the change of variables  $x = r \cos \theta$  and  $y = r \sin \theta$ , solve the integral

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy.$$

*Hint:* Calculate the *Jacobian* to do the change of variables, and then use a further substitution,  $u = -r^2$ , to simplify the integral that you get.

### 2. Neutron freeze-out

The difference in rest mass between a neutron and a proton is  $\Delta m = m_n - m_p = 1.293$  MeV, and the electron rest mass is  $m_e = 0.511$  MeV. The mass of the neutrino,  $m_\nu$ , is negligible in comparison.

- (a) The reaction  $n \rightleftharpoons p + e^- + \bar{\nu}_e$  occurs in equilibrium in the very early Universe, but becomes inefficient when  $k_B T \lesssim m_n - (m_p + m_e - m_\nu)$ .

Calculate the *neutron freeze-out* temperature,  $T_d$ , which is the temperature when the reaction first becomes inefficient. State your answer in Kelvin.

- (b) The equilibrium number densities of protons and neutrons,  $n_p$  and  $n_n$ , are related by

$$\frac{n_n}{n_p} = \left( \frac{m_n}{m_p} \right)^{\frac{3}{2}} \exp \left( -\frac{(m_n - m_p)c^2}{k_B T} \right).$$

Use your answer from part (a) to calculate the relative abundance of protons and neutrons immediately before freeze-out.

- (c) Briefly explain how the ratio  $n_n/n_p$  is expected to change between freeze-out and when all free neutrons become bound into nuclei.

### 3. Formation of the CMB

Briefly explain each of the following terms:

- (i) Recombination; (ii) Surface of last scattering; (iii) Decoupling.

#### 4. Distance to the surface of last-scattering

Consider a flat, matter-only Universe with  $H_0 = 70$  km/s/Mpc.

- Derive an expression for the comoving distance travelled by light that was emitted at a scale factor  $a$  that reaches us today at  $a = 1$ .
- Calculate the comoving distance to the surface of last scattering if decoupling occurred at  $z_{\text{dec}} = 1090$ .
- Calculate the angular diameter distance,  $d_A$ , and luminosity distance,  $d_L$ , to the surface of last scattering.
- Is the surface of last scattering within the comoving Hubble radius today,  $r_{\text{HR}}(a = 1)$ ?

#### 5. Energy density of CMB radiation **▲**

The spectral energy density (energy per unit volume per unit frequency) of a thermal (blackbody) gas of photons at temperature  $T$  is given by

$$u(\nu, T) = \frac{8\pi h}{c^3} \frac{\nu^3}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}.$$

- By integrating  $u(\nu, T)$  over frequency  $\nu$ , show that the total energy density of a thermal photon gas is

$$\rho_\gamma(T) = \frac{8\pi^5 k_B^4}{15h^3 c^3} T^4.$$

*Hint:* Perform a change of variables and then use the standard integral  $\int_0^\infty x^3 (e^x - 1)^{-1} dx = \pi^4/15$ .

- Calculate the fractional energy density of CMB radiation today,  $\Omega_{\text{CMB}} \equiv \rho_{\text{CMB}}(t_0)/\rho_{\text{crit},0}$ , if its blackbody temperature is  $T = 2.725$  K and the value of the Hubble parameter is  $H_0 = 67.4$  km/s/Mpc.

*Hint:* To convert between a mass density and energy density, recall that  $E = mc^2$ . The correct order of magnitude of the final result is  $\Omega_{\text{CMB}} \sim 10^{-5}$ .

- The fractional energy density of matter today is measured to be  $\Omega_m = 0.315$ . Calculate the ratio of the energy densities of CMB photons and matter,  $\rho_{\text{CMB}}(z)/\rho_m(z)$ , at decoupling ( $z_{\text{dec}} \approx 1090$ ).
- The redshift of matter-radiation equality is measured to be  $z_{\text{eq}} = 3402$  in our Universe.

Assume that the only two types of radiation are CMB photons with  $T = 2.725$  K and a thermal gas of (massless) neutrinos with temperature  $T = T_\nu$ . Use the measured value  $z_{\text{eq}} = 3402$  and the result from part (a) to calculate a blackbody temperature  $T_\nu$  for the neutrinos.

#### Assessed question: Electron number density during recombination

The Saha equation is given by

$$\frac{x_e^2}{1 - x_e} \approx \frac{1}{n_p + n_H} \left( \frac{m_e k_B T}{2\pi\hbar^2} \right)^{\frac{3}{2}} \exp\left(-\frac{E_\infty}{k_B T}\right),$$

where the ionisation fraction  $x_e = n_e/(n_p + n_H)$  and  $E_\infty = 13.6$  eV.

- When the temperature of the Universe is  $T = 4100$  K, the total number density of hydrogen nuclei is  $n_{\text{tot}} \equiv n_p + n_H = 5.45 \times 10^9$  m<sup>-3</sup>. Use the Saha equation to calculate the value of  $x_e$  at this time.  
(*Hint: Calculate the numerical value of the right-hand side of the equation before doing any algebra.*)
- The total number density of hydrogen nuclei scales like  $n_{\text{tot}} \propto a^{-3}$  regardless of the ionisation state of the hydrogen. Use this fact to find an expression for  $n_{\text{tot}}$  as a function of temperature,  $T$ .
- Use your results from parts (a) and (b) to calculate the value of  $y \equiv x_e^2/(1 - x_e)$  at the following temperatures:  $T = 4600, 4400, 4200, 4000, 3800, 3600, 3400, 3200$  K.

Hence, plot  $x_e$  as a function of  $T$ .

(*Hint: You will find it much less tedious to use a spreadsheet or Python notebook to calculate the results and make the plot.*)

## Tutorial question 6: Photon-baryon ratio

The number density of photons in a blackbody radiation field is given by

$$n_\gamma = \frac{8\pi}{(hc)^3} \int_0^\infty \frac{E^2 dE}{\exp(E/k_B T) - 1},$$

where  $E$  is the photon energy and  $T$  is the blackbody temperature.

- Use the standard integral  $\int_0^\infty x^2 (e^x - 1)^{-1} dx = 2.404$  to simplify the expression for  $n_\gamma$ .
- Calculate  $n_\gamma$  for the CMB observed today, at a temperature of  $T \approx 2.7$  K. State your answer in  $\text{m}^{-3}$ .
- The ionisation fraction immediately before decoupling can be approximated by the Saha equation,

$$\frac{x_e^2}{1 - x_e} \approx \frac{1}{n_p + n_H} \left( \frac{m_e k_B T}{2\pi \hbar^2} \right)^{\frac{3}{2}} \exp\left(-\frac{E_\infty}{k_B T}\right),$$

where the ionisation fraction is defined as  $x_e = n_e / (n_p + n_H)$ , and  $m_e k_B / (2\pi \hbar^2) = 1.80 \times 10^{14} \text{ m}^{-2} \text{ K}^{-1}$ .

The redshift of decoupling is  $z_{\text{dec}} \approx 1090$ . By assuming a suitable value of  $x_e$ , estimate the total number density of hydrogen nuclei,  $n_p + n_H$ , at decoupling, in units of  $\text{m}^{-3}$ .

- Use your answer from part (c) to estimate the photo-baryon ratio,  $\eta \equiv n_\gamma / n_b$ , at redshift  $z = 0$ .
- How has the photon-baryon ratio changed since the time of baryogenesis?

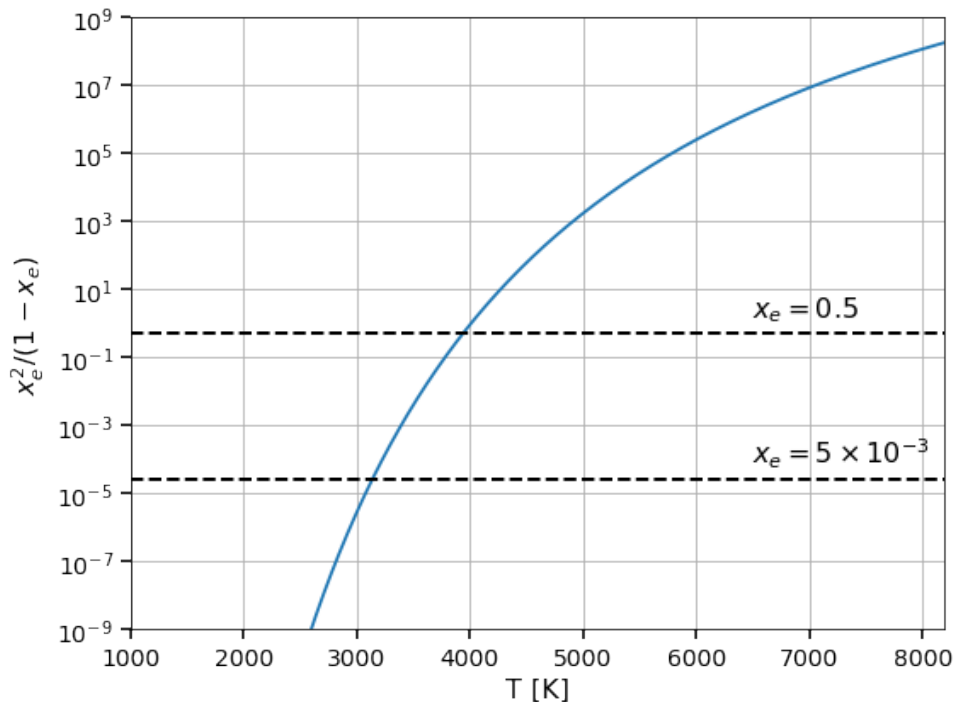


Figure 1: Plot showing how the ionisation fraction,  $x_e$ , changes with temperature.

## Section 8: CMB anisotropies

### 1. Maths practice: Spherical harmonics

- What are the possible values of  $m$  for the monopole ( $\ell = 0$ ), dipole ( $\ell = 1$ ), and 16-pole ( $\ell = 16$ )?
- Sketch the monopole and dipole modes on a Mollweide projection (choose any direction for the dipole).
- Show that the monopole and dipole modes are *orthogonal*, i.e. that

$$\int_0^\pi \int_0^{2\pi} Y_{0,0}(\theta, \phi) Y_{1,m}^*(\theta, \phi) \sin \theta \, d\theta d\phi = 0,$$

where  $*$  denotes the complex conjugate, and this expression should hold for any value of  $m$ . You can use the following definitions for the  $Y_{\ell,m}$  functions:

$$Y_{0,0} \propto 1; \quad Y_{1,0} \propto \cos \theta; \quad Y_{1,\pm 1} \propto \sin \theta e^{im\phi}.$$

### 2. Angular scales

- Calculate the approx. angular scales corresponding to (i)  $\ell = 10$ ; (ii)  $\ell = 100$ ; (iii)  $\ell = 1000$ , in degrees.
- A typical galaxy cluster has a physical size of about 1 Mpc. For a galaxy cluster at an angular diameter distance of 4.2 Gpc from us, what value of  $\ell$  does this correspond to?

### 3. Gravitational redshift

- Explain how gravitational potential wells on the surface of last scattering cause temperature anisotropies.
- Explain the difference between the Sachs-Wolfe and integrated Sachs-Wolfe effects.
- Explain how the integrated Sachs-Wolfe effect would change if there was no dark energy in our Universe.

### 4. Baryon acoustic oscillations

- By drawing a series of simple sketches, explain how the baryon acoustic oscillations arise.
- Describe how Doppler shifts can give rise to temperature anisotropies. Use this to explain why the power spectrum does not fall to zero in between the baryon acoustic oscillation peaks of the CMB power spectrum.

### 5. Angular power spectrum

- Sketch the power spectrum of the CMB as a function of spherical harmonic mode,  $\ell$ . Label the Sachs-Wolfe plateau, baryon acoustic oscillation peaks, and damping tail.
- How would your plot be affected if the surface of last scattering was further away from us (i.e. at a greater angular diameter distance)? (*Hint*: Assume that the comoving sizes of the acoustic peaks don't change.)

### 6. Designing a CMB experiment

Consider a telescope consisting of a microwave receiver attached to a parabolic dish. The angular resolution of the telescope depends on the observation wavelength  $\lambda$  and the diameter of the dish  $D$ , so that the smallest angular feature that it can resolve is  $\Delta\theta \approx \lambda/D$  (where  $\Delta\theta$  is in radians).

- What sized dish is needed to resolve features that are 1 degree across at an observing frequency of 70 GHz?
- The Wien Law tells us that the peak of a blackbody spectrum can be found at a wavelength of  $\lambda_{\text{peak}} = b/T$ , where  $b = 2898 \mu\text{m K}$  and  $T$  is the blackbody temperature. Assuming that our experiment observes at this wavelength, what sized dish does it need to resolve the CMB power spectrum out to  $\ell = 1000$ ?
- The South Pole Telescope observes at frequencies 90 GHz and 150 GHz, and has a dish that is 10m in diameter. What is the highest  $\ell$ -mode that it can observe in each frequency band?

**Assessed question: Gravitational redshift at last scattering**

Photons that are emitted in a gravitational potential well sustain an additional ‘gravitational redshift’, given by

$$\Delta z = \Delta\phi/c^2,$$

where  $\Delta z$  is the additional redshifting that occurs when a photon climbs out of a potential well of depth  $\Delta\phi = \phi(r_{\text{obs}}) - \phi(r_{\text{em}})$ , where  $r_{\text{obs}}$  and  $r_{\text{em}}$  are the positions of the observer and emission of the photon respectively.

- (a) Consider a spherically-symmetric potential well of shape  $\phi(r)/c^2 = 1 - \exp(-\frac{1}{2}(r/r_s)^2)$ . Sketch the potential well, making sure to label the axes. Restrict your plot to  $r$  values in the range  $[-5r_s, +5r_s]$ .
- (b) Consider an observer located at  $r = +5r_s$ . Draw a graph of the additional redshifting,  $\Delta z$ , of the photons seen by the observer, as a function of the position  $r$  that they were emitted at.
- (c) Derive an expression for the fractional change in the observed temperature of the photons,  $(\Delta T)/T_{\text{em}}$ , caused by the gravitational redshift,  $\Delta z$ . *Hint: Perform a Taylor expansion around  $\Delta z = 0$ .*
- (d) Use your expression from part (c) to explain what observations of CMB temperature anisotropies can tell us about the *shape* and *depth* of the gravitational potential on the last scattering surface.
- (e) **Bonus challenge question (not assessed) ▲**: The peculiar velocity of a small free-falling parcel of gas in the potential well is given by  $\vec{v} = -\nabla\phi$ . Plot the peculiar velocity as a function of  $r$ , and hence infer how the Doppler shift  $\Delta z$  depends on  $r_{\text{em}}$ .

## Tutorial question 7: First acoustic peak of the CMB power spectrum

ESA's Planck satellite has mapped the CMB temperature anisotropies to extremely high precision over almost the entire sky. As a result, it has been able to measure the CMB power spectrum with very small statistical uncertainties out to high  $\ell$ . This includes making the most precise measurement possible of the first peak of the CMB.

- Briefly explain the physical origin of the first acoustic peak in the CMB power spectrum.
- What is an approximate numerical value of the comoving sound horizon at last scattering,  $r_s$ , in Mpc?
- Using data from the Planck satellite, we can infer a *comoving* sound horizon size at last-scattering of  $r_s(z_*) = 144.46 \pm 0.48$  Mpc, and an angular diameter distance to the last scattering surface of  $d_A = 12.72$  Mpc. The redshift to last-scattering is measured to be  $z_* = 1090.30 \pm 0.41$ . Use this information to calculate the angular size of the sound horizon,  $\theta_s$ , in degrees, and its angular scale,  $\ell_s$ .
- Measure the location of the first acoustic peak,  $\ell_*$ , from the Planck power spectrum below.
- Compare your measurement from part (d) with the theoretical value that you calculated in part (c).
- Consider a scenario in which the acoustic peak had been observed at  $\ell_s = 180$  instead. Assuming that the redshift of last scattering and the size of the comoving sound horizon are unchanged, calculate how this would change the angular diameter distance to last scattering.

Hence, suggest how the location of the first peak can be used to precisely measure cosmological parameters.

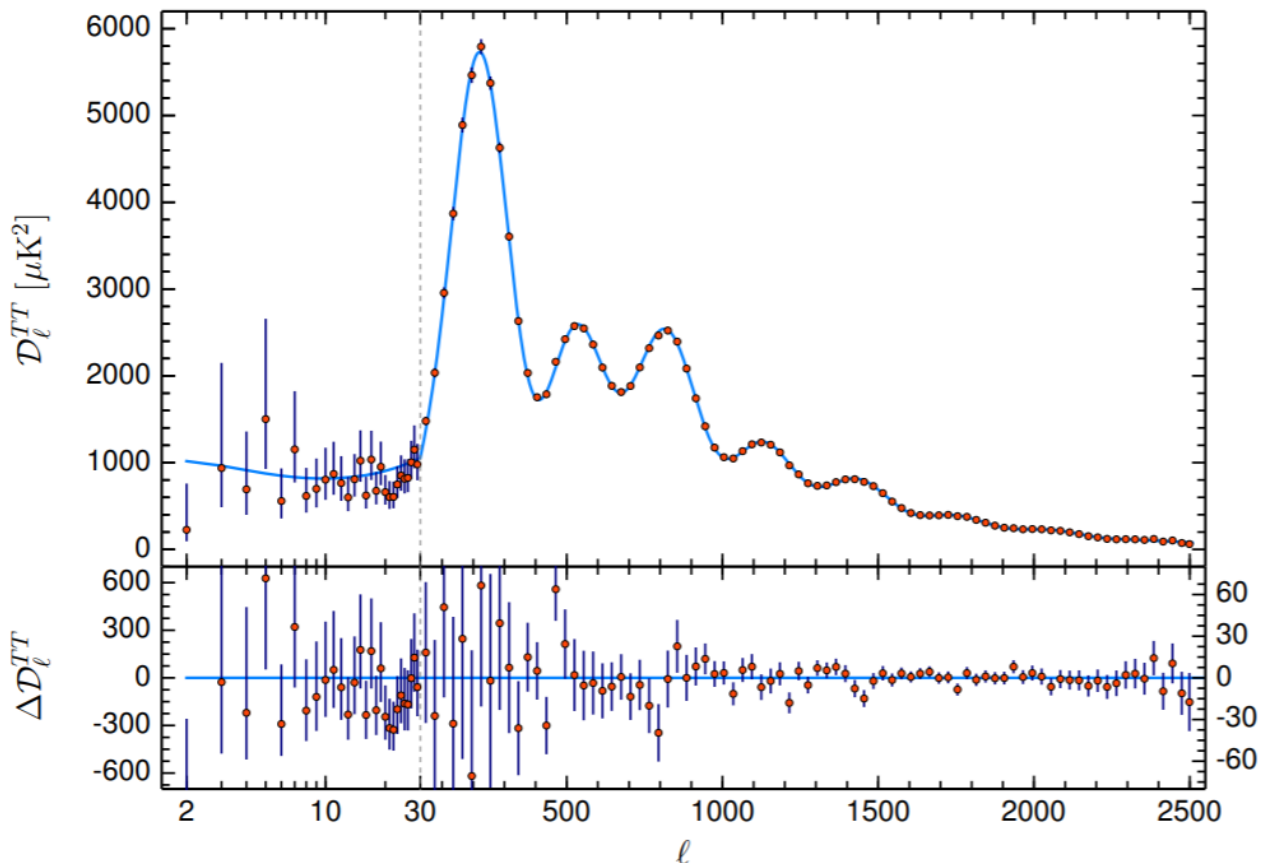


Figure 2: The angular power spectrum of temperature anisotropies, as measured by the Planck satellite. The upper panel shows the measured power spectrum,  $\mathcal{D}_\ell \propto \ell(\ell+1)C_\ell$ , and the best-fit theoretical model. The lower panel shows the difference between the data and the best-fit model. Note that the x-axis has a non-linear scale, which goes 10, 20, 30, 100, 200... (Credit: ESA/Planck).

## Section 9: Inflation

### 1. Maths practice: Neglecting small terms

- By performing a Taylor expansion, expand  $y = x(1+x)^{-2}$  to linear order in the small parameter  $x \simeq 0$ .
- A damped harmonic oscillator has an equation of motion  $\ddot{x} + 2b\omega\dot{x} + \omega^2x = 0$ . We will assume that the damping is positive,  $b > 0$ , but relatively weak,  $b \lesssim \omega$ .
  - Using the same notation as above, write down the equation of motion for a *simple* harmonic oscillator.
  - Sketch graphs of  $\dot{x}$  and  $\ddot{x}$  as a function of position,  $x$ , for a simple harmonic oscillator.
  - By considering how  $\ddot{x}$ ,  $\dot{x}$ , and  $x$  behave as a function of  $x$  for a simple harmonic oscillator, suggest which of the terms in the equation of motion for the damped harmonic oscillator can be neglected when (i)  $x \approx 0$ ; (ii)  $x \approx \pm x_{\max}$ .
  - How would your answers change if the damping was very strong,  $b \gg \omega$ ?

### 2. Slow-roll approximation

The cosmological Klein-Gordon equation for a scalar field is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0.$$

- Use the Friedmann equation to find an expression for  $H^2$  in a flat universe containing only a scalar field  $\phi$ .
- Show that  $H^2 \approx (8\pi G/3)V(\phi)$  in the slow-roll approximation.
- Write down the Klein-Gordon equation in the slow-roll approximation.
- You should notice that one quantity that was neglected in the Friedmann equation still remains in the Klein-Gordon equation. Explain why this is.

### 3. The inflaton as a ‘clock’ field

If the inflaton field only travels in one direction (i.e.  $\dot{\phi}$  is always positive or always negative), we can use it as a time coordinate. This is analogous to the way that we use the scale factor  $a$  as a time coordinate because it increases monotonically in an expanding universe.

Using the slow-roll approximation of the cosmological Klein-Gordon equation, find an expression relating the infinitesimal time interval  $dt$  to the infinitesimal field displacement  $d\phi$ .

### 4. Hubble radius during inflation

Sketch how the Hubble radius  $r_{\text{HR}}(a)$  depends on the scale factor during slow-roll inflation. Use your sketch to explain how inflation solves the horizon and monopole problems.

### 5. Flatness problem

If the Universe is exactly flat, the total fractional energy density parameter  $\Omega_0 = 1$ .

- Show that  $|1 - \Omega_0| \propto (aH)^{-2}$ .
- Consider a Universe that is very slightly negatively curved today,  $\Omega_k = 10^{-3}$ , and that is otherwise radiation-dominated ( $\Omega_r \approx 1$ ). Evaluate  $1 - \Omega_0$  at:
  - $a = 0.1$ ;
  - $a = 10^{-3}$ ;
  - $a = 10^{-10}$  for this Universe.
- Inflation ended when the Universe was at a temperature of  $T \approx 10^{22}$  K. Assuming that  $T_0 = 2.725$  K today, calculate the approximate scale factor corresponding to the end of inflation.
- Use your answers above to briefly explain what the flatness problem is, and how inflation solves it.

### 6. Scale factor during inflation **A**

Consider a scalar field with potential  $V(\phi) = e^{\lambda\phi}$  that is the only matter/energy field in a flat, expanding universe.

- Write down the Klein-Gordon and Friedmann equations for this cosmological model under the slow-roll approximation. Make sure to substitute in the definition of  $V(\phi)$  from above.
- Solve these equations to find an expression for the scalar field as a function of the scale factor,  $\phi(a)$ . You may use units where  $8\pi G = 1$ , and the initial condition  $\phi(a_i) = \phi_i$ .
- Consider a situation in which the inflaton moves a distance  $\Delta\phi = 0.6$  between the beginning and end of inflation. Calculate the ratio of the final to initial scale factors,  $a_f/a_i$ , for this model if  $\lambda = -0.01$ .
- Inflation is thought to have ended when the Universe reached a temperature of  $T \approx 10^{22}$  K. Calculate how many e-folds of expansion have occurred between the end of inflation and today (when  $T \approx 2.7$  K).
- Calculate the value of  $\Delta\phi$  that would be required for as many e-folds of expansion to have happened *during* inflation as have happened *since* inflation in this inflationary model, assuming that  $\lambda = -0.01$ .

#### Assessed question: The end of inflation

Consider a universe in which the energy density at very early times is dominated by a slowly-rolling inflaton field, with energy density  $\rho_\phi(a_i) = \rho_i$  at an initial scale factor of  $a_i$ . For convenience, we will denote the expansion rate at this time as  $H_* = \sqrt{8\pi G\rho_i/3}$ .

- By first solving the Friedmann equation to find  $a(t)$ , derive an expression for the duration of inflation,  $\Delta t$ , if inflation ends after 60 e-folds of expansion, at a scale factor  $a_f$ .
- Sketch how the Hubble radius,  $r_{\text{HR}}(a)$ , changes as a function of scale factor over these 60 e-folds of expansion. Make sure the label the axes. (*Hint*: Use logarithmic axes for your plot!)
- Assume that reheating happens instantaneously at  $a_f$ , with the inflaton field decaying into radiation with the same energy density. The Universe then becomes completely radiation-dominated. Calculate how many e-folds of expansion during the radiation-dominated era would be needed to undo the change in  $r_{\text{HR}}(a)$  that occurred during inflation.  
*Hint*: Calculate the ratio of the Hubble radius with its value at the start of inflation,  $r_{\text{HR}}(a_i)$ .

## Tutorial question 8: Exponential potential

Consider a scalar field  $\phi$  that has a potential of the form  $V(\phi) = Ae^{\lambda\phi}$ .

- Sketch the potential  $V(\phi)$  for  $A = 1$  and  $\lambda = 1$ , including positive and negative values of  $\phi$ .  
On the same sketch, show how  $V(\phi)$  would change if you increased the values of (i)  $A$ , and (ii)  $\lambda$ .
- Write expressions for the kinetic energy, pressure, and density of the scalar field in terms of  $\dot{\phi}$  and this potential (or its derivatives).
- Consider initial conditions for the scalar field  $\phi_i = 1$ ,  $\dot{\phi}_i = 0$ . Describe the qualitative behaviour of the kinetic energy, pressure, density, and equation of state of the scalar field immediately after it starts to move. Which direction will the field begin to move in?



Figure 3: Inspiration for while you're thinking about scalar fields. (Credit: J. Huang/CC-BY-2.0)

## Section 10/11: Dark matter and structure formation

### 1. Dark matter

- Name three pieces of observational evidence for dark matter.
- Explain the difference between particle dark matter and baryonic dark matter.
- Define *hot dark matter*, *warm dark matter*, and *cold dark matter*.
- Explain how the halo mass function differs between the WDM and CDM scenarios. Make reference to the mean free path of dark matter particles in your answer.

### 2. Temperature of dark matter

Consider a dark matter particle with mass  $m$  that is in thermal equilibrium with radiation until it freezes out.

- Consider a universe with a radiation temperature today of  $T \approx 3$  K. Write down an expression for  $T_{\text{freeze}}$ , the radiation temperature at the redshift that the dark matter particle freezes out,  $z_{\text{freeze}}$ .
- Write down an inequality for the dark matter mass if it is to be considered *cold* dark matter from the time of freeze-out onwards.
- The momentum of the dark matter particle scales with scale factor  $\propto 1/a$ . Find an approximate expression for the scale factor at which a particle of any mass would become non-relativistic.

### 3. Galaxy correlation function

- Briefly explain what the galaxy correlation function,  $\xi_g(r)$ , tells us about pairs of galaxies.
- Write down the relationship between the matter correlation function and the galaxy correlation function. Define the parameter  $b$  in this expression.
- Sketch the galaxy correlation function. Label your sketch with the approximate position of the BAO feature, in Mpc.

### 4. Definition of the power spectrum

The matter power spectrum,  $P(k)$ , is defined through the expression  $\langle \delta(\vec{k})\delta^*(\vec{k}') \rangle = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') P(k)$ .

- Define the quantity  $\delta(\vec{k})$  in this expression.
- Briefly explain what the angle brackets  $\langle \dots \rangle$  mean.
- Show that the correlation function  $\xi(r) \equiv \langle \delta(\vec{x})\delta^*(\vec{x} + \vec{r}) \rangle$  is the Fourier transform of the power spectrum. *Hint:* Substitute the expression for the Fourier transform of  $\delta(\vec{x})$ , and then use the definition of the power spectrum from above. The angle brackets commute with the Fourier integrals.

### 5. Density-velocity cross-spectrum

In Fourier space, the relationship between the peculiar velocity  $v$  and density contrast  $\delta$  can be written as

$$v(\vec{k}, a) = ia \frac{\dot{D}}{D} \frac{\delta(\vec{k}, a)}{k}$$

- Use the definition of the growth rate,  $f$ , to rewrite this expression as  $v(\vec{k}, a) = iaHf \delta(\vec{k}, a)/k$ .
- The cross-correlation between the density contrast and velocity is defined as

$$\langle v(\vec{k}, a)\delta^*(\vec{k}', a) \rangle = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') P_{v\delta}(k),$$

where  $P_{v\delta}(k)$  is the *density-velocity cross-spectrum*. Show that  $P_{v\delta}(k) = iaHfP(k)/k$ .

### 6. Growth rate of structure $\blacktriangle$

The growth rate,  $f$ , is defined as

$$f(a) = \frac{d \log D}{d \log a}.$$

Consider a scenario in which  $f = \alpha$ , where  $\alpha$  is a constant.

- (a) Integrate the equation for the linear growth rate to find an expression for  $D(a)$ , subject to the initial condition  $D = 1$  at  $a = 1$ .
- (b) By inserting an appropriate expression for the expansion rate,  $H(a)$ , find a simplified expression for the growth equation,

$$\frac{d}{da} \left( a^3 H(a) \frac{dD}{da} \right) = \frac{3}{2} \frac{H_0^2 \Omega_m a^{-3}}{H^2(a)} a H(a) D(a),$$

under the assumption of a flat, matter-only universe.

- (c) Insert your solution for  $D(a)$  into the simplified growth equation. Use the result to show that  $\alpha = 1$  in a flat, matter-only universe.
- (d) There is another solution for  $\alpha$  that satisfies the flat, matter-only growth equation. Find this value, and sketch how  $D(a)$  varies with scale factor in this case.
- (e) Comment on how the density contrast would change with time in this case, and suggest why we usually only consider the  $\alpha = 1$  solution to the growth equation.

**Assessed question: Density fluctuations**

Consider a 1D matter density field inside a box that extends from  $x = 0$  to  $x = L$ . The density at position  $x$  and scale factor  $a$  is given by  $\rho_m(x, a) = \bar{\rho}_0 a^{-3} [1 + 0.1a \sin(8\pi x/L)]$ .

- (a) Write down an expression for the density contrast in the box,  $\delta$ , as a function of position,  $x$ , and scale factor,  $a$ .
- (b) Calculate the value of the density contrast at (i) the bottom of the deepest void; and (ii) the peak of the most over-dense region, both at  $a = 1$ .
- (c) Calculate an expression for the average density in the box,  $\langle \rho_m \rangle = (1/L) \int_0^L \rho_m(x, a) dx$ , at scale factor  $a$ . Hence, show that  $\langle \delta \rangle = 0$  for any scale factor  $a$ .

## Revision Questions I

### Section 1: Expanding universe

1. An emission line has a rest-frame wavelength of 1240 nm. The same emission line is observed in a distant galaxy at a wavelength of 1377 nm. What is the redshift of the galaxy?
2. The true cosmological redshift of a galaxy is  $z = 0.014$ . Its peculiar velocity is 720 km/s in the direction away from us. Calculate the *observed* redshift of the galaxy.
3. Calculate the recession velocity of a galaxy with a true cosmological redshift of  $z = 0.34$  and a peculiar velocity of 402 km/s in the direction towards us.
4. Explain how the Doppler shift affects the wavelength of light. Make sure to explain the sign of the effect.
5. State Olbers' paradox. How is it resolved?
6. The expansion rate today is  $H_0 \approx 70$  km/s/Mpc. If a galaxy of initial diameter 42 kpc was expanding with the Hubble flow, how much bigger would it be after 1000 years?
7. Write down an expression for the observed redshift  $z$  of an object with cosmological redshift  $z_c$  and peculiar velocity  $v$ .
8. What is the difference between the recession velocity and peculiar velocity of an object?
9. Show that typical galaxy peculiar velocities ( $v \lesssim 1000$  km/s) are an important contribution to the observed redshift for nearby (low-redshift) objects, but are negligible at high redshifts.

### Section 2: Geometry and distance

1. State the Hubble Law. Explain why the Hubble Law is predicted by an expanding universe.
2. What is the parallax of an object at a distance of 102 pc? Give your answer in arcsec.
3. Explain the difference between parallax angle and the angular size of an object.
4. Explain the difference between proper and comoving coordinates.
5. If an object stays at an exactly constant comoving distance from us as the Universe expands, what does this say about its peculiar velocity?
6. Write down the line element for a flat FLRW metric in spherical polar coordinates. What is the meaning of  $ds$ ?
7. State the three different types of spatial curvature that a homogeneous space can have.
8. For each type of spatial curvature, state the sign of  $k$  and the sign of  $\Omega_k$ . Write down inequalities for the sum of angles in a triangle and the total matter/energy density (compared to critical density) in each case.
9. Calculate the Taylor expansion of (i)  $\sin x$ ; (ii)  $\cos x$ ; and (iii)  $e^x$  about  $x = 0$ . Give your answers to linear order in the expansion.

### Section 3: Friedmann equation

1. Write down the Friedmann equation.
2. Define the expansion rate  $H$  in terms of time derivatives of the scale factor.
3. Write down an expression for  $\rho_\Lambda$ , the energy density of the cosmological constant  $\Lambda$ .
4. Calculate the expansion rate at a redshift  $z = 1.4$  in a flat, matter-only Universe with  $H_0 = 67$  km/s/Mpc.

5. Consider a flat universe containing only matter and a cosmological constant. The expansion rate at three redshifts is given by:  $H(z = 0.1) = 73.54$  km/s/Mpc;  $H(z = 0.7) = 96.85$  km/s/Mpc;  $H(z = 1.4) = 138.79$  km/s/Mpc. Use these data to calculate the values of  $\Omega_\Lambda$  and  $H_0$ .
6. Write down the definition of the critical density today,  $\rho_{\text{cr},0}$ .
7. Write down the definition of  $\Omega_m$  in terms of the critical density today and the matter density  $\rho_m(t)$ .
8. Divide the Friedmann equation by the critical density. Use your result to prove that  $\Omega_m + \Omega_r + \Omega_k + \Omega_\Lambda = 1$  in a universe that contains matter, radiation, curvature, and a cosmological constant.
9. Define  $\Omega_0$ . How is  $\Omega_0$  related to  $\Omega_k$ ?
10. Write down the definition of  $\Omega_k$  in terms of the curvature constant,  $k$ .
11. Calculate  $\Omega_k$  for a universe where  $\rho_{\text{tot}}(t_0) = 0.93\rho_{\text{cr},0}$ .
12. Calculate the critical density of a universe with  $H_0 = 102$  km/s/Mpc. State your answer in units of  $\text{kg/m}^3$ .
13. Consider a universe that is made up of 12% massless neutrinos, 28% CMB photons, and 14% cold dark matter. The remainder is made up of a cosmological constant. What is  $\Omega_r$  in this universe?
14. Write down how the densities of matter, radiation, and a cosmological constant vary with scale factor,  $a$ .
15. Solve the Friedmann equation to find  $a(t)$  for a flat, matter-only universe with initial conditions  $a = 0$  at  $t = 0$ . Calculate the age of this universe,  $t_0$ , in Gyr.
16. Solve the Friedmann equation to find  $a(t)$  for a flat, radiation-only universe with initial conditions  $a = 0$  at  $t = 0$ . Sketch your solution,  $a(t)$ .
17. Solve the Friedmann equation to find  $a(t)$  for an empty universe with negative curvature, and initial conditions  $a = 0$  at  $t = 0$ . Sketch your solution,  $a(t)$ .
18. Solve the Friedmann equation to find  $a(t)$  for a flat, cosmological constant-only universe with initial conditions  $a = 0$  at  $t = 0$ . If  $H_0 = 82$  km/s/Mpc in this universe, what will be the value of the scale factor when it reaches an age of 10 Gyr?
19. Explain the differing fates of a matter-only universe that is (i) flat; (ii) open; (iii) closed.
20. Find expressions for the following quantities in terms of the expansion rate,  $H$ , and scale factor,  $a$ : (i)  $da/dt$ ; (ii)  $da/dz$ ; (iii)  $dz/dt$ ; (iv)  $dk/dt$  (where  $k$  is the curvature parameter).

#### Section 4: Distances and horizons

1. Write down the definition of the luminosity distance in terms of the observed flux  $f$  and intrinsic luminosity  $L$  of an object.
2. Write down the definition of the angular diameter distance of an object in terms of its angular size  $\Delta\theta$  and physical (proper) diameter  $d$ .
3. Is the angle  $\Delta\theta$  in the definition above in degrees or radians?
4. Explain the difference between the parallax distance and angular diameter distance.
5. Write down definitions of the luminosity distance and angular diameter distance in terms of the comoving distance travelled by light,  $r(z)$ , and redshift  $z$ .
6. If a galaxy at redshift  $z = 3.83$  lies at an angular diameter distance of  $d_A = 1460$  Mpc, what is its luminosity distance (in Mpc)? What is the comoving distance travelled by light from this galaxy?
7. A galaxy of proper (physical) diameter 41 kpc is observed to subtend an angle of 1.8 arcsec. What is the angular diameter distance to the galaxy?

8. A galaxy has an angular size of 1.5 arcsec at a redshift  $z = 0.5$ . Calculate its physical (proper) diameter in a matter-only universe with  $H_0 = 71$  km/s/Mpc.
9. Calculate the proper (physical) diameter of a galaxy that is observed with an angular size of 4.1 arcsec at an angular diameter distance of  $d_A = 104$  Mpc.
10. Explain the terms *standard candle* and *distance ladder*.
11. A nearby galaxy contains a Cepheid variable star with known luminosity  $L = 2890 L_\odot$ , observed with flux  $f = 4.5 \times 10^{-3} L_\odot/\text{Mpc}^{-2}$ . Calculate the luminosity distance to the galaxy.
12. Use the FLRW line element to find an integral expression for the comoving distance travelled by light in a flat universe with arbitrary expansion rate  $H(a)$ .
13. Derive an expression for the comoving distance travelled by light,  $r(z)$ , in a flat, matter-only universe.
14. Find expressions for the following quantities in terms of the expansion rate,  $H$ , and scale factor,  $a$ : (i)  $dr/dt$ ; (ii)  $dr/da$ ; (iii)  $dr/dz$ .
15. Define the term *particle horizon*. Write down an expression for the particle horizon in any flat FLRW universe.
16. Write down a general expression for the Hubble radius,  $r_{\text{HR}}(a)$ .
17. Calculate the Hubble radius for (i) a matter-only universe; (ii) a radiation-only universe; (iii) a cosmological constant-only universe. You may assume flatness.
18. Sketch the Hubble radius as a function of scale factor in the three universes from the question above.

### Section 5: Cosmic acceleration

1. Write down the equation of state parameters for (i) matter; (ii) radiation; (iii) a cosmological constant; (iv) massless relativistic neutrinos; (v) baryons; (vi) cold dark matter.
2. Use the conservation equation to show that a type of mass/energy with constant density ( $\rho(t) = \text{const.}$ ) has an equation of state  $w = -1$ .
3. Consider a flat universe with  $\Omega_m = 0.3$ ,  $\Omega_\Lambda = 0.7$ , and  $H_0 = 60$  km/s/Mpc. Use the Raychaudhuri equation to find the scale factor at which the expansion of this universe first starts to accelerate ( $\ddot{a} > 0$ ).
4. Show that  $d(\dot{a}^2)/dt = 2\dot{a}\ddot{a}$ .
5. Calculate the time derivative of  $y = (\dot{a}/a)^2$ .
6. Explain why the following definition cannot be correct:  $q(a) = -(1 + \dot{H}/H)$ .
7. The deceleration parameter is given by  $q(a) = -(1 + \dot{H}/H^2)$ . Show that this leads to the following alternative definition:
 
$$q(a) = -\left(\frac{a}{\dot{a}}\right)^2 \frac{\ddot{a}}{a}. \quad (1)$$
8. Calculate the deceleration parameter today,  $q_0$ , in (i) a matter-only universe; (ii) a cosmological constant-only universe; (iii) an empty universe with negative curvature ( $k < 0$ ).
9. Solve the Friedmann equation to find  $a(t)$  for a flat, cosmological constant-only universe. Use your solution to show that  $H = \text{const.}$
10. Sketch the Hubble radius as a function of scale factor in a flat, cosmological constant-only universe.
11. Briefly explain the *cosmological constant problem*.
12. Using the Raychaudhuri equation, conservation equation, or otherwise, derive a relation between  $q$  and  $w$ .
13. Use the conservation equation to find how the density depends on scale factor for types of mass/energy with equation of state (i)  $w = 0$ ; (ii)  $w = -2/3$ ; and (iii)  $w = +1$ .

## Revision Questions II

### Section 7: CMB radiation

1. Write down how temperature depends on redshift.
2. Define the terms *decoupling*, *recombination*, and *last scattering*.
3. How does the mean-free path of photons change at last scattering?
4. Explain the process of decoupling.
5. Would decoupling have happened earlier or later in the Universe's history if the number density of electrons,  $n_e$ , had been larger?
6. Calculate the approximate energy, in eV, of a photon with temperature  $T = 4000$  K.
7. Explain the process of recombination.
8. Explain why recombination happens at a lower energy than 13.6 eV (the ionisation energy of hydrogen).
9. Define the *last-scattering surface*.
10. What would be the redshift of the last-scattering surface if the CMB had been emitted at a temperature of  $T = 40,000$  K? (The observed CMB temperature today is  $T_0 = 2.725$  K.)
11. How would a Doppler shift due to the peculiar velocity of our galaxy affect the observed CMB temperature?

### Section 8: CMB anisotropies

1. Find the approximate angular size, in degrees, of CMB anisotropies observed at  $\ell = 300$ .
2. Briefly explain what is meant by a *temperature anisotropy* in the CMB radiation?
3. Explain how regions of the last scattering surface with higher than average density affect the observed CMB temperature.
4. Explain how the velocity of matter at the last scattering surface affects the CMB temperature.
5. Explain the difference between the Sachs-Wolfe and integrated Sachs-Wolfe effects. Which is a primary anisotropy, and which is a secondary anisotropy?
6. Sketch the CMB power spectrum as a function of spherical harmonic wavenumber,  $\ell$ . Mark the approximate position of the acoustic peak.
7. On your sketch of the CMB power spectrum, label the features caused by the Sachs-Wolfe effect, integrated Sachs-Wolfe effect, and baryon acoustic oscillations.
8. Explain why the CMB power spectrum drops as  $\ell$  increases.
9. Explain how the baryon acoustic oscillations formed.
10. Explain how the integrated Sachs-Wolfe effect provides evidence for cosmic acceleration.
11. Explain how the baryon acoustic oscillations provide evidence for the existence of dark matter.

### Section 9: Inflation

1. Explain the horizon problem. Refer to at least one piece of observational evidence in your answer.
2. Explain the flatness problem. Refer to at least one piece of observational evidence in your answer.
3. Explain the monopole problem. Refer to at least one piece of observational evidence in your answer.
4. Show that a universe with  $w = -1$  leads to exponential expansion,  $a(t) \propto e^{Ht}$ .
5. Show that  $H \approx \text{const.}$  in an exponentially-expanding universe.
6. Show that the Hubble radius decreases with time in an exponentially-expanding universe.
7. Sketch the Hubble radius in (i) an exponentially-expanding universe, and (ii) a radiation-only universe, as a function of scale factor.
8. Explain how inflation solves the horizon and monopole problems.
9. Explain how inflation solves the flatness problem.
10. Write down expressions for the energy density and relativistic pressure of a scalar field,  $\phi$ .
11. Show that, in the slow-roll approximation, the equation of state of the scalar field is  $w_\phi \approx -1$ .
12. Write down the Klein-Gordon equation in the slow-roll approximation. Justify why you have kept or neglected each term.
13. Write down the Friedmann equation in the slow-roll approximation, in a flat universe containing only a scalar field.
14. Sketch a potential,  $V(\phi)$ , that would allow slow-roll inflation to occur.

### Section 10: Dark matter

1. Name three pieces of observational evidence for dark matter.
2. Explain why each piece of evidence from the question above implies the existence of hidden (invisible) mass.
3. State three known properties of dark matter.
4. Explain the difference between particle dark matter and baryonic dark matter.
5. Name one piece of evidence that disfavors baryonic dark matter.
6. Explain the difference between hot dark matter, warm dark matter, and cold dark matter.
7. Explain how the temperature of dark matter affects the formation of structures.
8. Describe one piece of evidence that favors CDM over WDM.

### Section 11: Structure formation

1. Briefly define each kind of large-scale structure: halos/clusters, voids, filaments.
2. Write down the (mathematical) definition of the density contrast,  $\delta$ .
3. Define the growth factor,  $D(a)$ , in terms of the density contrast at different times.
4. Define the growth rate,  $f$ , in terms of derivatives of the growth factor.
5. Show that  $D \propto a$  is a solution to the growth equation if the universe contains only matter.
6. Calculate  $f(a)$  if  $D \propto a$ .

7. Briefly explain how the expansion of space affects the growth of structure.
8. Define the matter power spectrum,  $P(k)$ , in terms of the variance of the density contrast.
9. Evaluate  $\int \delta^{(3)}(\vec{x} - \vec{x}') f(\vec{x}) g(\vec{x}') d^3x$ .
10. Sketch the correlation function,  $\xi(r)$ . Label the baryon acoustic oscillation feature, including its approximate location in Mpc.
11. Explain how the galaxy correlation function,  $\xi_g(r)$ , depends on the galaxy bias,  $b$ .
12. Explain what the correlation function tells us about the distribution of galaxies.